**ND Gaussian Integrals**

Now we’ll generalize to N dimensional integrals.

**N Dimensional real Gaussian Integrals**

Now we’ll consider an N dimensional integral (real), or rather its moment generating function (Einstein summation convention employed):



[we can make this 3N if we want, and the xi can run over all N particles’ 3 coordinates – whatever] We can work out its value. We’ll assume **A** is a symmetric matrix. Therefore it is diagonalizable. Let its eigenvectors be Uij and its eigenvalues be a. Then A = U(a)UT. Aij = Uiα(aα)Ujα. And xiAijxj = xiUiα(aα)Ujαxj. Let yα­ = xiUiα, and equivalently, yαUjα = xiUiαUjα → yαUjα = xj. Then we can write this as: yα(aα)yα. Also note: Aij-1 = Uiα(1/aα) Ujα. The Jacobian transformation would be 1, because matrix is unitary. So we have:



So we find the natural generalization of our 1D result



Still, we find the saddle point rule. Consider:



So we can write:



Moments, again, are given by Taylor series:



These moments aren’t as easily evaluated in complete generality as they were in 1D. And the cumulant generating function is W(**j**) = lnZ(**j**) of course:



And in our case we have:



So we see that only the second cumulant will survive. And now we’ll explore representing correlations/averages diagrammatically. Consider for example the following correlation:



This could’ve been inferred from the moment generating function anyway. But this establishes **A**-1 as the ‘propagator’. We can extend this in parallel with the previous section to calculate integrals of the form,



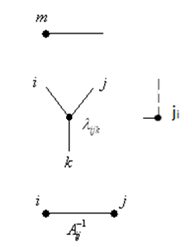
where Z(0) is just, Z evaluated with j, λ set to zero. And the indices m…n don’t all have to be different – we can an x raised to a power. We can say, formally,



So we have:



Don’t confuse the mp’s with the m and n summation index. We can get the Dmn’s with Feynman diagrams. Our parts would then change only slightly. Aij-1 is the propagator, and jℓ, λijk, would be the vertices. And then we would sum over internal vertices.



And after constructing all topologically distinct Feynman diagrams (FD) we’d have:



where p is the total number of external points/legs. And the Multiplicity is the number of ways to construct such a topologically distinct diagram from the parts. And the Symmetry Factor is, well Multiplicity/(3!)nm!n!. As we did in 1D, we should still have that:



The Feynman rules for the symmetry factors are:



This presumes that λijk is an index-independent scalar, λ. This is the overwhelming case of interest anyway.

**N dimensional complex number Gaussian integrals**

Now let’s move on to complex numbers. The moment generating function is:



This can be worked out, again. We’ll assume **A** is a Hermitian matrix. Therefore it is diagonalizable. Let its eigenvectors be Uij and its eigenvalues be *a*i. Then A = U(a)U†. Aij = Uiα(aα)Ujα\*. And zi\*Aijzj = ziUiα(aα)Ujα\*zj. Let wα­ = ziUiα, and equivalently, wαUjα\* = ziUiαUjα\* → wαUjα\* = zj. Then we can write this as: wα(aα)wα. Also note: Aij-1 = Uiα(1/aα) Ujα\*. The Jacobian transformation would be 1, because matrix is unitary. So we have:



So we have:



Ass usual, we’ll find we can write:



The Taylor series expansion of the moment generating function gives moments of course:



The cumulant generating function is given by W(**j**,**j**\*) = lnZ(**j**,**j**\*), as usual:



and in our case we have:



And let’s consider the diagrammatic representation of some correlations. First off:



This establishes **A**-1 as the ‘propagator’. Higher order cumulants follow Wick’s theorem. For instance,



We can generalize this to:



So we have:



Don’t confuse the mp’s and nq­’s with the m and n summation indices. We can get the Dmm´n’s with Feynman diagrams. Our parts would then change only slightly. Aij-1 is the propagator, and jℓ, λijkl, would be the vertices. And then we would sum over internal vertices.

A picture containing sky, flock, wire

Description automatically generated

And we can write Dmm´n as:



where the Multiplicity is the number of ways to construct such a diagram from the parts. And the Symmetry Factor is, well Multiplicity/(2!2!)nm!m´!n!. As in 1D, we should still have that for any particular diagram Dmm´n, the sum of the multiplicities of every Feynman diagram we can write for it is:



The Feynman rules for the symmetry factors are:



This presumes, I’d imagine, that λ is symmetric in all indices, so a scalar.

**N-dimensional Grassman integrals**

And we can do more complex integrals. Let’s consider something like,



with implicit Einstein summation over repeated indices of course. And dNdNψ = d1dψ1 d2dψ2…dNdψN. Going to look at Z(0,0) first, again.



Screw it. I’m just going to specialize to 2 dimensional matrix.



So going to generalize and say,



And going to take on not too much faith at this point that:



As usual, we should also find that:



Now let’s consider a 2D correlation,



since, recalling elementary inverse of 2×2 matrix – have to Google it every time.



So though we could probably prove this in general doing a change of variables thing like we did for complex variables, I’ll take this example as proof enough that:



And if we did a more complicated one, should get:



Notice the minus sign coming from permutations required to put the given ψ, right next to each other when doing the ‘contraction’. So we can clearly see Wick’s theorem at work here. We can generalize this to (I don’t feel like changing the \* to so deal with it),



So we have:



Don’t confuse the mp’s and nq­’s with the m and n summation indices. We can get the Dmm´n’s with Feynman diagrams. Our parts would then change only slightly. Aij-1 is the propagator, and jℓ, λijkl, would be the vertices. And then we would sum over internal vertices.

A picture containing sky, flock, wire

Description automatically generated

And we can write Dmm´n as:



where the Multiplicity is the number of ways to construct such a diagram from the parts. And the Symmetry Factor is, well Multiplicity/(2!2!)nm!m´!n!. As in 1D, we should still have that for any particular diagram Dmm´n, the sum of the multiplicities of every Feynman diagram we can write for it is:



The Feynman rules for the symmetry factors are:



This presumes, I’d imagine, that λ is symmetric in all indices, so a scalar.

**Fermion Loops**

But there is one more rule to consider having to do with sign issues arising from having to permute the ψ’s in the proper *ψ* to pull out the contraction. And this pertains to Fermion loops.



where ε = ±1 for Fermions/Bosons. The diagrams below have 3, 4, 3 Fermion loops, I think.

Shape

Description automatically generated

And then like before, we sum over all internal indices.