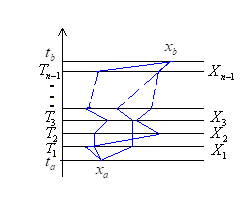
**Path Integrals**

I’m mainly interested in path integrals vis a vis partition functions and correlations/green’s functions. But we’ll skip all that here and just give the facts…

**Continuously infinite real variable Gaussian integrals**

Now let’s consider translating to the continuum. Formally, take a finite interval t ∈ [a,b], and divide it up into N parts, taking the limit that N → ∞. Let each X(T) vary between (-∞,∞), but with restriction that X(ta) = xa and X(tb) = xb.



Then, the moment generating function for our distribution would be:



Note we write D[x(t)] to indicate that the differential is sort of a functional of the path, in that:



with caveat that beginning and end of path is xand x´ respectively. And formally, in analogy with the diiscrete case, we find that:



Despite the fact that the formula for the result is easy to state, it doesn’t seem that it is easy to calculate. First, we have to define the inverse operator, G(t,t´) = A-1(t,t´). It’s defined via analogy with matrix multiplication:



Then what about the pre-factor Z[0]? It seems that attempting to evaluate 1/√det(A/2π) is especially tricky. One possibility is to perform the following manipulations. Let the operator A have eigenvalues aj. Then,



Then use,



where *a* is the diagonal matrix of eigenvalues, and U is matrix of eigenvectors, ,and we use the cyclic property of trace. So we have:



Borrowing a little QM notation, we could write this as:



But this sum doesn’t seem to always/ever converge. Another possibility is to appropriate our results from QM and simply say:



I don’t *think* this expression is well defined unless some boundary conditions are specified on the operator/kernel A(s,t). Or in other words, unless we specify what x(t) must be at ta and tb. These boundary conditions would usually be evident from the physical problem its describing, like propagation of a particle from x(ta) = xa to x(tb) = xb. If no boundary conditions are specified, then it’d seem that we’re just presuming a = -∞, b = ∞, and that x(t) goes to 0 at the end points. Then maybe the e(1/2)∫dSdT j(S)·G(S,T)·j(T) factor can be worked out explicitly with A-1 in hand. But it can also be done via the sp approximation as usual. And that might be the easier route. To demonstrate the equivalence, consider:



where we’ve assumed the operator A(t,t´) is symmetric. Now solving for x(t), we apply the inverse operator A-1 to both sides of our equation:



and then plugging it into our ‘action’, we have:



So it works out as we can see. And it usually seems that evaluating S[xsp(t)] by directly plugging in xsp(t) is easier than by attempting to use this last expression here. Either way, then we have:



With this we can get moments via functional differentiation. And then we have the cumulant generating function per usual W[j(t)] = lnZ[j(t)]:



Derivatives of this will give us cumulants. Whatever. Now for correlations, and their representation:



Now suppose we want to calculate:



This will be:



We can get the Dmn’s with Feynman diagrams. Our parts are a natural generalization of the ND Gaussian integrals parts. A-1(s,t) is the propagator, and j(t), λ(t,s,r), would be the vertices. And then we would integrate over internal times.

Diagram

Description automatically generated

And after constructing all topologically distinct Feynman diagrams (FD) we’d have:



where p is the total number of external points/legs. And the Multiplicity is the number of ways to construct such a topologically distinct diagram from the parts. And the Symmetry Factor is, well Multiplicity/(3!)nm!n!. As we did in 1D, we should still have that:



The Feynman rules for the symmetry factors are:



This presumes that λ(t,s,r) is constant scalar, λ. This is the overwhelming case of interest anyway.

**Equal Time Issue**

We also have another rule that pertains to time-ordering. We have/will have been parameterizing the real, complex, grassman variables as x(t), z(t), ψ(t). And if t is the independent variable, then the inverse operator is often ambiguously defined for a lot of operators A-1(s,t). So when we have legs connected to the same vertex, then s = t. But A-1(t,t) is undefined I think, so we interpret this as A-1(t,t+), where t+ is slightly larger than t.

***Example***

Now let’s consider such functional integrals defined over a 3D volume.



To better compare with our previous expressions, we might want to write this as:



The result would be:



Again, we’d need boundary conditions on φ, i.e., we’d need to know what φ(x) equals along the perimeter of the points under consideration, I guess (this is the idea I’m invoking with the φa, φb notation). But often we just presume to take the boundary out to infinity and for φ → 0 there. What is G = A-1 in this case? We’d recognize it as following from:



Often this is written,



But despite the notation, must not think that A-1(x,z) is actually proportional to a delta function. We can go to the momentum basis and easily invert. Just take Fourier transform of equation,



Taking inverse FT, we have:



Filling this into our Z, we’d have:



We can attempt to determine the pre-factor Z[0] for what it’s worth. We need the eigenvalues of the A operator. So we need to work out,



Well such eigenvectors are ψ(x) = eikx, with eigenvalue (k2 + m2). So we have:



We could technically compute this integral. If we’re dealing with a finite system, then it’d converge to something. If we’re dealing with an infinite system then we have to use manipulating to get a result. There’s another formal manipulation via which we could’ve obtained this result. Might want to do it this way if we can’t necessarily find the eigenvalues.



This is as far as we can go. But if we plug in a resolution of identity using the operators eigenstates, we can work this out more. Going to get rid of the explit integral expression for the Tr though,



Well off by a factor of (2π)d – whatever. In the continuum limit, this would seem to diverge for large k, i.e., in the ultraviolet limit. Whatevs. Of course there should be an ultra-violet cutoff k ~ 2π/a. A calculation of this quantity is performed in the Stat Mech folder/GF, Z Path Integrals Formulation file. Now say we want to calculate:



Feynman rules would be:

Diagram

Description automatically generated

And we’d integrate over all internal coordinates within the volume. Note the φ3 term would really have to be expressed as ∫∫∫φ(x)φ(y)φ(z)δ(x-y)δ(y-z), and it is the δ functions which tell us that all its legs are evaluated at the same point (x), and that we just integrate over x (because the two δ functions collapse the other two integrals). And of course then we would integrate over all internal coordinates. Also of course, it is advantageous to go to momentum space to carry out any integrations.

**Continuously infinite complex variable Gaussian integrals**

Generalizing our real functional distribution to a complex functional distribution, we have:



Generically, the solution is:



Boundary conditions at t = ta, tb seem necessary. And as usual we would expect:



Let’s consider an elementary correlation.



Now suppose we want to calculate something like:



This will be:



We can get the Dmm´n’s with Feynman diagrams. Our parts are a natural generalization of the ND complex Gaussian integrals parts. A-1(s,t) is the propagator, and j(t), j\*(t), λ(t,s,r,v), would be the vertices.

A picture containing bird, flock, wire

Description automatically generated

And then we integrate over all internal times/positions. And after constructing all topologically distinct Feynman diagrams (FD) for a given m, m´, n, we’d have:



where the Multiplicity is the number of ways to construct such a diagram from the parts. And the Symmetry Factor is, well Multiplicity/(2!2!)nm!m´!n!. As in 1D, we should still have that for any particular diagram Dmm´n, the sum of the multiplicities of every Feynman diagram we can write for it is:



The Feynman rules for the symmetry factors are:



This presumes, I’d imagine, that λ(t,r,s,v) is a constant, λ.

**Equal Time Issue**

We also have another rule that pertains to time-ordering. We have/will have been parameterizing the real, complex, grassman variables as x(t), z(t), ψ(t). And if t is the independent variable, then the inverse operator is often ambiguously defined for a lot of operators A-1(s,t). So when we have legs connected to the same vertex, then s = t. But A-1(t,t) is undefined I think, so we interpret this as A-1(t,t+), where t+ is slightly larger than t.

**Continuously infinite Grassman variable Gaussian integrals**

Finally, proceeding to Grassman integrals, we have:



Generically, the solution is:



Boundary conditions at t = ta, tb seem necessary. And as usual we would expect:



Let’s consider an elementary correlation.



Now suppose we want to calculate something like (using *ψ*\* and synonymously):



This will be:



We can get the Dmm´n’s with Feynman diagrams. Our parts are a natural generalization of the ND complex Gaussian integrals parts. A-1(s,t) is the propagator, and j(t), j\*(t), λ(t,s,r,v), would be the vertices.

A picture containing bird, flock, wire

Description automatically generated

And then we’d integrate over all internal times. And after constructing all topologically distinct Feynman diagrams (FD) for a given m, m´, n, we’d have:



where the Multiplicity is the number of ways to construct such a diagram from the parts. And the Symmetry Factor is, well Multiplicity/(2!2!)nm!m´!n!. As in 1D, we should still have that for any particular diagram Dmm´n, the sum of the multiplicities of every Feynman diagram we can write for it is:



The Feynman rules for the symmetry factors are:



This presumes, I’d imagine, that λ is a constant.

**Fermion Loops**

But there is one more rule to consider having to do with sign issues arising from having to permute the ψ’s in the proper *ψ* to pull out the contraction. And this pertains to Fermion loops.



where ε = ±1 for Fermions/Bosons. The diagrams below have 3, 4, 3 Fermion loops, I think.

Shape

Description automatically generated

**Equal Time Issue**

We also have another rule that pertains to time-ordering. We have/will have been parameterizing the real, complex, grassman variables as x(t), z(t), ψ(t). And if t is the independent variable, then the inverse operator is often ambiguously defined for a lot of operators A-1(s,t). So when we have legs connected to the same vertex, then s = t. But A-1(t,t) is undefined I think, so we interpret this as A-1(t,t+), where t+ is slightly larger than t.