**Vector Path Integrals**

Now going to add another/final layer of complexity. Our integration variable could itself be a vector, and we’ll want to explore integrations over all of its possible *components’* values in space and/or time.

**Continuously infinite real vector variable Gaussian integrals**

We can generalize our continuous variables to vectors too. The moment generating function would be:



where,



where x1,2,3 are the x,y,z components of the vector. And **x**·**A**·**x** means ΣijxiAijxj. Similarly with **j**·**x**. Again boundary conditions at t = ta and t = tb would seem necessary. Formally, the solution would be:



And as usual, we expect



and I think we can say,



And correlations for instance, would be given by:



Now say we want to calculate correlations like this (implicit summation over repeated indices):



This will be:



We can get the Dmn’s with Feynman diagrams. Our parts are a natural generalization of what’s gone before.

Diagram

Description automatically generated

And then we would integrate over internal times, and sum over all internal indices. And after constructing all topologically distinct Feynman diagrams (FD) we’d have:



where p is the total number of external points/legs. And the Multiplicity is the number of ways to construct such a topologically distinct diagram from the parts. And the Symmetry Factor is, well Multiplicity/(3!)nm!n!. As we did in 1D, we should still have that:



The Feynman rules for the symmetry factors are:



This presumes that λlmn(t,s,r) is constant scalar, λ.

**Equal Time Issue**

We also have another rule that pertains to time-ordering. We have/will have been parameterizing the real, complex, grassman variables as xi(t), zi(t), ψi(t). And if t is the independent variable, then the inverse operator is often ambiguously defined for a lot of operators A-1mn(s,t). So when we have legs connected to the same vertex, then s = t. But

Amn-1(t,t) is undefined I think, so we interpret this as Amn-1(t,t+), where t+ is slightly larger than t.

**Internal Traces**

Also, if we have a two legs of a vertex connected to each other, like we have below (4 times)

Diagram

Description automatically generated with medium confidence

then typically, this is automatically, typically, a trace of the Green’s function,



Usually the diagonal components of the GF = Amm-1 are identical, and so this just gives

n×Amm-1(t,t+), where n is the number of components of the vector **x**. Anyway, I’ve written about it more in other files where we have taken a more explicit look at GF’s and the rules for calculating them.

***Example***

Consider the following example (Einstein summation over repeated indices)



The solution should be:



Need boundary conditions as cited before. The inverse operator would be, in this case:



Result should be as usual,



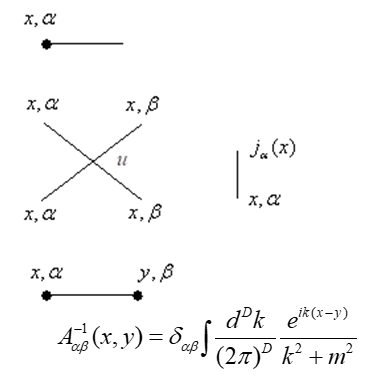
Now say we wanted to evaluate something like:



where (Einstein summation again),



It isn’t really harder to do this. We simply generalize the x index into an (x,α) index which labels both points and components. Our Feynman rules would therefore be, in this case,



and we’d sum over all internal coordinates (within the volume) and vertices. Now consider a more complicated interaction vertex,



The last two terms are straightforward –we’ve already done them. The first term varies from the above 4-vertex only in that one each of the α and β legs would carry a on it, and this would act on the GF it rides on.

**Continuously infinite complex vector variable Gaussian integrals**

And we can generalize to continuous complex variables to vectors too. The moment generating function would be:



And solution is:



Boundary conditions at t = ta, tb would seem necessary. And



A correlation, for instance, would be:



Now suppose we want to calculate something like (implicit summation over repeated indices):



This will be:



We can get the Dmm´n’s with Feynman diagrams.

Diagram

Description automatically generated

And then we integrate over all internal times/positions. And after constructing all topologically distinct Feynman diagrams (FD) for a given m, m´, n, we’d have:



where the Multiplicity is the number of ways to construct such a diagram from the parts. And the Symmetry Factor is, well Multiplicity/(2!2!)nm!m´!n!. As in 1D, we should still have that for any particular diagram Dmm´n, the sum of the multiplicities of every Feynman diagram we can write for it is:



The Feynman rules for the symmetry factors are:



This presumes, I’d imagine, that λijkl(t,r,s,v) is a constant, λ.

**Equal Time Issue**

We also have another rule that pertains to time-ordering. We have/will have been parameterizing the real, complex, grassman variables as xi(t), zj(t), ψk(t). And if t is the independent variable, then the inverse operator is often ambiguously defined for a lot of operators A-1mn(s,t). So when we have legs connected to the same vertex, then s = t. But

Amn-1(t,t) is undefined I think, so we interpret this as Amn-1(t,t+), where t+ is slightly larger than t.

**Internal Traces**

Also, if we have a two legs of a vertex connected to each other, like we have below (4 times)

Diagram

Description automatically generated with medium confidence

then typically, this is automatically, typically, a trace of the Green’s function,



Usually the diagonal components of the GF = Amm-1 are identical, and so this just gives

n×Amm-1(t,t+), where n is the number of components of the vector **x**. Anyway, I’ve written about it more in other files where we have taken a more explicit look at GF’s and the rules for calculating them.

**Continuously infinite Grassman vector variable Gaussian integrals**

And last we have (where dot products are interpreted as matrix multiplication):



Generically, the solution is:



Boundary conditions at t = ta, tb seem necessary. And as usual we would expect:



Let’s consider an elementary correlation.



Now suppose we want to calculate something like (using *ψ*\* and synonymously):



This will be:



We can get the Dmm´n’s with Feynman diagrams.

Diagram

Description automatically generated

And then we integrate over all internal times/positions. And after constructing all topologically distinct Feynman diagrams (FD) for a given m, m´, n, we’d have:



where the Multiplicity is the number of ways to construct such a diagram from the parts. And the Symmetry Factor is, well Multiplicity/(2!2!)nm!m´!n!. As in 1D, we should still have that for any particular diagram Dmm´n, the sum of the multiplicities of every Feynman diagram we can write for it is:



The Feynman rules for the symmetry factors are:



This presumes, I’d imagine, that λijkl(t,r,s,v) is a constant, λ.

**Fermion Loops**

But there is the fermion loop thing to consider.



where ε = ±1 for Fermions/Bosons. The diagrams below have 3, 4, 3 Fermion loops, I think.

Shape

Description automatically generated

**Equal Time Issue**

We also have another rule that pertains to time-ordering. We have/will have been parameterizing the real, complex, grassman variables as xi(t), zj(t), ψk(t). And if t is the independent variable, then the inverse operator is often ambiguously defined for a lot of operators A-1mn(s,t). So when we have legs connected to the same vertex, then s = t. But

Amn-1(t,t) is undefined I think, so we interpret this as Amn-1(t,t+), where t+ is slightly larger than t.

**Internal Traces**

Also, if we have a two legs of a vertex connected to each other, like we have below (4 times)

Diagram

Description automatically generated with medium confidence

then typically, this is automatically, typically, a trace of the Green’s function,



Usually the diagonal components of the GF = Amm-1 are identical, and so this just gives

n×Amm-1(t,t+), where n is the number of components of the vector **x**. Anyway, I’ve written about it more in other files where we have taken a more explicit look at GF’s and the rules for calculating them.

***example***

Can consider a few standard cases from Condensed Matter. So from many body physics, the standard Hamiltonian is:



and GFC\* is:



where,



and K = H – μN. The Feynman diagram parts would be:

Diagram

Description automatically generated

We haven’t had two-leg diagrams so far, because we’ve presumed that we had diagonalized the entire two-leg interaction. But if that’s too hard (we *could* solve for the GF’s which include V1, but we usually don’t) then we can include the pesky two-leg part as a perturbation, just as we did with the one-leg j(x) terms. Also, I guess there’d be a ½ attached to the V2(x-x´)δ(t-t´) actually, but this will go away, when multiplicities are taken account of, namely the equivalence of x and x´ – see Stat Mech/GF’s files, among others). Won’t get into the symmetry factors and all, as this case is also covered in Stat Mech/GF’s files. The symmetry rules are different than those quoted above, as those are for constant λ interactions, not x-dependent λ interactions. The propagator is given by:



with implicit periodic/anti-periodic boundary conditions on imaginary time variable over interval (0, β). We can represent as Fourier transform,



where ωn is 2π(n+1/2)/β if fermions and 2πn/β if bosons. And we can plug this into the PDE,



Can see we need,



to reproduce the RHS. Turns out this is actually the negative of the GF (well, see definition above), as commonly defined for thermally averaged systems (see Stat Mech/GF). So we have:



We can put this in position/time space if we want, but of course once we construct the diagrams, we usually evaluate them in momentum/energy space instead. Also, if we go back to those files in Stat Mech folder, we’ll see that the expansion for the GF is most naturally written in terms of -G, which is our A-1, a coincidence that makes sense in light of our path integral formulation.

***example***

Now consider a T = 0 Green’s function based of the Hamiltonian.



and GFC is (see QM/Many Identical Particles/GF Path Integral Formulation):



where,



Putting things in the standard form of our Path integrals exp( -two\_leg\_interactions + everything else), we’d have the following Feynman diagram parts,

Text, letter

Description automatically generated

(again, the ½ isn’t attached to V2 for multiplicity of switching x and x´) Won’t get into the symmetry factors and all, as this case is also covered in QM/Multiple Identical Particles/GF’s files. The symmetry rules are different than those quoted above, as those are for constant λ interactions, not x-dependent λ interactions. The propagator is given by:



From the aforementioned files, we know that the PDE for the (unperturbed) causal GF is:



and identifying A-1 = iG matches this. And this accords with our formula for GC up above. This explains why the diagrammatic expansion is most naturally written in terms of iGc.